6 Array Effective Directivity Index with Gaussianly Perturbed Signal Wavefront.

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ARRAY EFFECTIVE DIRECTIVITY INDEX WITH GAUSSIANLY PERTURBED SIGNAL WAVEFRONT

by

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INTRODUCTION

In the design of large arrays, the question arises of possible directivity index degradation due to a randomly perturbed signal wavefront. In this memorandum, the array effective directivity index is found for such a situation as a function of the number of array elements, N, and average interelement correlation functions, for signal and noise sound fields. Developments are given for the cases of independent perturbations and Markov-Gaussian correlated perturbations.

DEVELOPMENT

Let $s_i(t)$ be the voltage output of the i^{th} omnidirectional element of an N element array, in the signal acoustic field. Each element may be assumed to be compensated (in time delay) for the non-perturbed wavefront. Due to a perturbation imposed on the wavefront during its passage through the medium, however, the output of the i^{th} hydrophone differs in delay from that of the reference hydrophone by an amount τ_i. For any hydrophone, τ_i is assumed normally distributed with variance σ_τ^2 about the mean corresponding to the ideal wavefront.

We may thus represent the summed voltage output from the array for signal only, $s_T(t)$, as

$$s_T(t) = \sum_{i=1}^{N} s_i(t) = \sum_{i=1}^{N} s_0(t-\tau_i) \qquad (1)$$

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where $s_0(t)$ = reference hydrophone output in the signal field.

The mean-square signal voltage, then, is:

$$\overline{s_T^2(t)} = \sum_{i,j=1}^{N} \overline{s_0(t-\tau_i)\cdot s_0(t-\tau_j)}$$

$$= \overline{s_0^2(t)}\left[N + \sum_{\substack{i,j=1\\ i\neq j}}^{N} {}_s\varphi_0(\tau_i-\tau_j)\right] \qquad (2)$$

where ${}_s\varphi_0(\tau)$ = reference hydrophone normalized autocorrelation function for the signal field.

and the bar indicates time average.

It will be noted that a stationary time series is assumed for the hydrophone outputs. If it is further assumed that the probability of occurrence of the perturbation τ_i at the i^{th} hydrophone is independent of the perturbation at all the other hydrophones, then the following definition of the average signal interelement correlation function, $\overline{\varphi_s}$, has meaning:

$$\overline{\varphi_s} = \frac{1}{N(N-1)} \sum_{\substack{i,j=1\\ i\neq j}}^{N} \varphi_0(\tau_i-\tau_j) \qquad (3)$$

Substituting from (3) into (2), then, gives:

$$\overline{s_T^2(t)} = \overline{s_0^2(t)}\left[N + N(N-1)\overline{\varphi_s}\right] \qquad (4)$$

An exactly similar derivation may be made for the noise field (no properties of either field have yet been assumed), yielding:

$$\overline{n_T^2(t)} = \overline{n_0^2(t)}\left[N + N(N-1)\overline{\varphi_N}\right] \qquad (5)$$

The effective directivity index, EDI, is related to the increased discrimination against noise of an array versus an omnidirectional hydrophone:

$$\boxed{EDI = 10\log_{10}\left\{\left(\frac{\overline{A_T^2(t)}}{\overline{n_T^2(t)}}\right)\Big/\left(\frac{\overline{A_0^2(t)}}{\overline{n_0^2(t)}}\right)\right\}} \quad (6)$$

From (4) and (5) into (6), then

$$EDI = 10\log_{10}\left\{\frac{1+(N-1)\overline{\overline{\varphi}}_s}{1+(N-1)\overline{\overline{\varphi}}_N}\right\} \quad (7)$$

If the average interelement correlation function for the noise field is very small, as it is in most cases of interest, then

$$\boxed{EDI \approx 10\log_{10}\left\{1+(N-1)\overline{\overline{\varphi}}_s\right\}} \quad (8)$$

RELATION OF $\overline{\overline{\varphi}}_s$ TO PERTURBATION STATISTICS

$\overline{\overline{\varphi}}_s$ is still not in a form useful for the desired computations. If N is a large number and the perturbations are independent, then we may replace the space (or ensemble) average $\overline{\overline{\varphi}}_s$ by the statistical average, $\tilde{\varphi}_s$, where

$$\tilde{\varphi}_s = \int_{-\infty}^{\infty} {}_s\varphi_0(\Delta\tau)\cdot p(\Delta\tau)\,d(\Delta\tau) \quad (9)$$

and $\Delta\tau = \tau_i - \tau_j$ (any $i \neq j$)

This is an ergodic property of the wavefront which is quite reasonable to expect under the conditions we have assumed.

Let $p(\tau_i)$, for any i, be

$$p(\tau_i) = (2\pi\sigma_\tau^2)^{-\frac{1}{2}} \exp\left\{-\frac{\tau_i^2}{2\sigma_\tau^2}\right\} \tag{10}$$

Then, under our independence assumption,

$$p(\Delta\tau) = p(\tau_i - \tau_j) = (4\pi\sigma_\tau^2)^{-\frac{1}{2}} \exp\left\{-\frac{(\Delta\tau)^2}{4\sigma_\tau^2}\right\} \tag{11}$$

Substituting from (11) into (9),

$$\overline{{}_s\varphi_o} = \overline{\varphi_s} = (4\pi\sigma_\tau^2)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \exp\left\{-\frac{(\Delta\tau)^2}{4\sigma_\tau^2}\right\} {}_s\varphi_o(\Delta\tau)\, d(\Delta\tau) \tag{12}$$

Now, for single frequency or narrow band, we find (if $\omega\tau \leq \pi$)

$${}_s\varphi_o(\Delta\tau) \approx \cos(\omega\Delta\tau) \tag{13}$$

From (13) into (12), then

$$\overline{{}_s\varphi_o} \approx (4\pi\sigma_\tau^2)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \exp\left\{-\frac{(\Delta\tau)^2}{4\sigma_\tau^2}\right\} \cos\omega\Delta\tau\, d(\Delta\tau) \tag{14}$$

The average interelement normalized correlation function is thus the real-variable moment-generating function of the perturbation distribution. For the Gaussian case at hand, from (14),

$$\bar{s}_{\phi}^{\varphi} \approx e^{-\omega^2 \sigma_\tau^2} \tag{15}$$

Substituting from (15) into (8), gives

$$\boxed{EDI \approx 10 \log_{10} \left\{ 1 + (N-1) e^{-\omega^2 \sigma_\tau^2} \right\}} \tag{16}$$

or, for single frequency, since $\phi = \omega\tau$, $\sigma_\phi = \omega\sigma_\tau$, and

$$EDI \approx 10 \log_{10} \left\{ 1 + (N-1) e^{-\sigma_\phi^2} \right\} \tag{17}$$

Several limiting cases may be examined. First, if σ_τ^2 approaches 0, then

$$EDI \Big|_{\sigma_\phi^2 \to 0} \longrightarrow 10 \log N \tag{18}$$

Secondly, if σ_τ^2 approaches infinity, then the last term on the right in (16) approaches zero, so the EDI approaches zero in this case, Since having σ_ϕ^2 go to infinity corresponds to complete destruction of the signal wavefront, this answer is intuitively satisfying.

In Figure 1, the EDI is plotted versus σ_ϕ for various values of N from 32 to 32,000. When the product

$$(N-1) e^{-\sigma_\phi^2} \gg 1,$$

the db loss in EDI due to σ_τ is independent of N. This occurs for the higher N's and lower σ_{τ}^{2}/a, toward the left of the figure.

CORRELATED PERTURBATIONS

Wavefront perturbations correlated from hydrophone to hydrophone over the whole array create essential complications. It is possible to attack such a problem using first-order Markov theory as follows:

Let us assume the $(i+1)^{th}$ perturbation is related to the i^{th} perturbation thus:

$$\tau_{i+1} = \rho \tau_i + \sqrt{1-\rho^2}\, y_{i+1} \quad , \; -1 \leq \rho \leq 1 \tag{19}$$

where ρ is a constant and y_{i+1} is a variable whose variance is the same as τ_i, but which is random with respect to τ_i. It is easily shown that ρ is the normalized cross correlation between τ_{i+1} and τ_i both of which possess the same variance. The argument may be extended to show that ρ^n is the normalized cross-correlation between the i^{th} and the $(i+n)^{th}$ perturbations. For the Gaussian case (governed by a second-order Gaussian distribution, the so-called Markov-Gaussian condition) it can be easily shown, using a method suggested by B. F. Cron, that the distribution of the difference between the i^{th} and the $(i+n)^{th}$ perturbations is Gaussian with mean zero and variance ${}_n\sigma_\Delta^{2}$ given by

$$ {}_n\sigma_\Delta^{2} = \sigma_\tau^{2} \left[(1-\rho^{n})^{2} + (1-\rho^{2n}) \right] \tag{20}$$

As a check, we see that for $\rho = 0$, ${}_n\sigma_\Delta^{2}$ is twice σ_τ^{2} and independent of n, which is the assumption used to derive (16) and (17). For $\rho = 1$, ${}_n\sigma_\Delta^{2}$ is zero, which means that we have again an ideal wavefront with no perturbations, and so no degradation in EDI.

To calculate the effect on EDI, let us start from (2). We must replace (3) by a series of equations, resulting for the second term of (2), in

$$\sum_{\substack{i,j=1 \\ i \neq j}}^{N} {}_s\varphi_o(\tau_i - \tau_j) = 2 \sum_{n=1}^{N-1} (N-n) \cdot \overline{{}_s\varphi_o\{(\Delta\tau)_n\}} \tag{21}$$

where $(\Delta\tau)_n$ indicated the difference of perturbations from elements n elements apart.

Now, since we still have to do with Gaussian distributions, as in (14), but ones in which the variances are different, we can immediately write down from (15) that

$$
\begin{aligned}
{}_s\varphi_0^{new}\{\Delta\tau_n\} &= e^{-\frac{\omega^2{}_n\sigma_\Delta^2}{2}} \\
&= e^{-\omega^2\{\sigma_\tau^2\}\{(1-\rho^n)^2+(1-\rho^{2n})\}/2}
\end{aligned} \tag{22}
$$

Substituting from (22) into (21) and (2) then gives:

$$
\overline{A_r^2(t)} = \overline{A_0^2(t)}\left[N + 2\sum_{n=1}^{N-1}(N-n)e^{-\omega^2\{\sigma_\tau^2\}\{(1-\rho^n)^2+(1-\rho^{2n})\}/2}\right] \tag{23}
$$

and, with the assumption of negligible noise correlation (if this is not true, an equation similar to (7) must be used), there finally results, for the case of correlated perturbations,

$$
\boxed{EDI \approx 10\log_{10}\left[1 + 2\sum_{n=1}^{N-1}\left(1-\frac{n}{N}\right)e^{-\omega^2\{\sigma_\tau^2\}\{(1-\rho^n)^2+(1-\rho^{2n})\}/2}\right]} \tag{24}
$$

where ρ^n is the normalized cross-correlation between perturbations from elements n apart, N is the number of array elements, and σ_τ^2 is the variance of the perturbation distribution for (any) one element output.

Interest has been expressed in obtaining an expression like (24), from which the EDI may be developed for specific assumptions. No further discussion will be given here, other than to indicate that when a MacLaurin series approximation to the exponential is permissible, computational difficulties can probably be eased.

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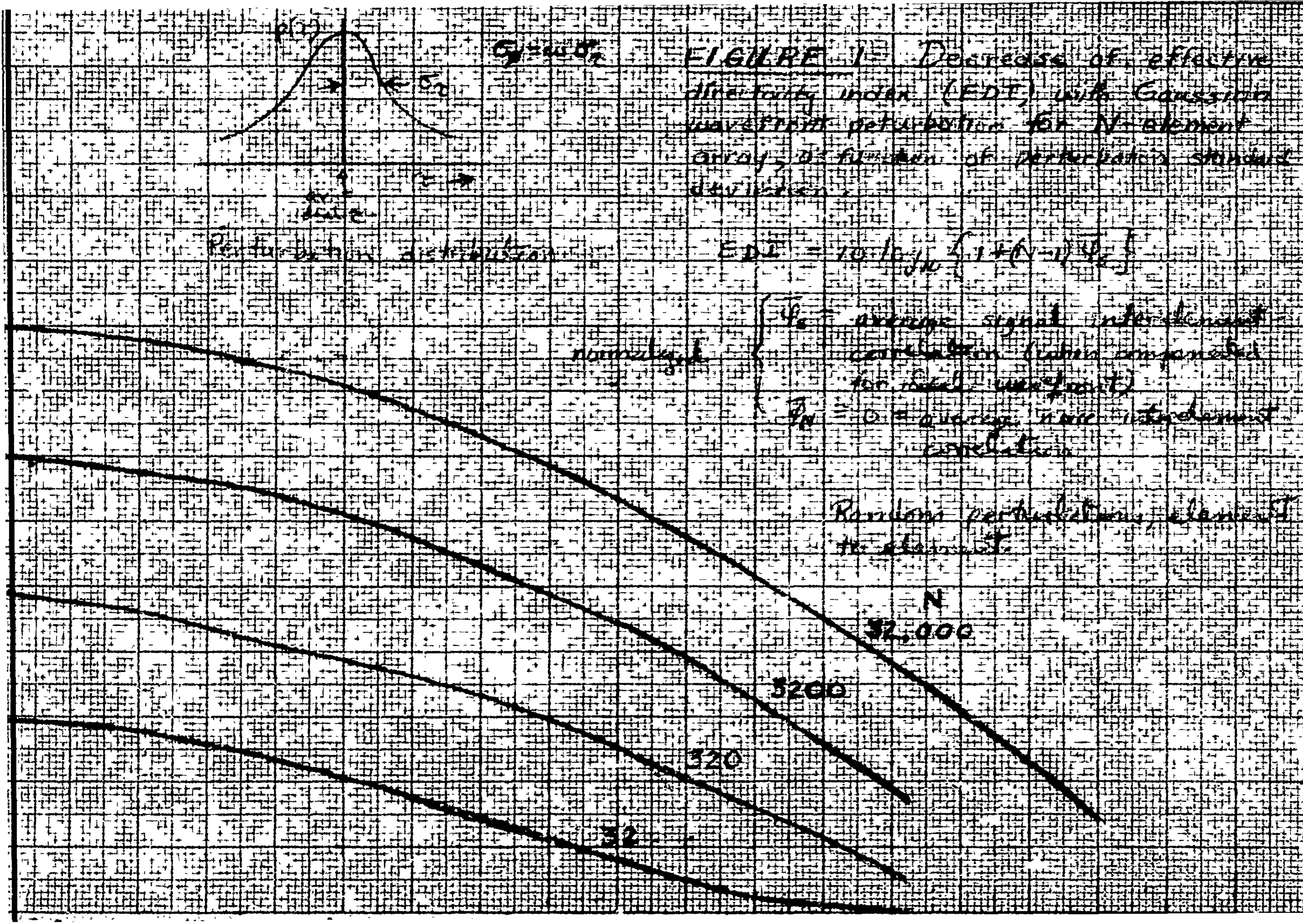

FIGURE 1 Decrease of effective directivity index (EDI) with Gaussian wavefront perturbation for N-element array, σ = function of perturbation standard deviation.